A multifactor, stochastic volatility HJM model in a low dimensional markov representation: theory overview and implementation details

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A stochastic volatility extended Cheyette Model: theory overview and implementation details

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YASM
Yet Another Shortrate Model

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Contents

Introduction

We have the LMMs why bother with shortrate models anyway?

- looking for a slim alternative to LMM
- try to capture as many features as possible with only few state variables
- few state variables will be faster in MC simulation
- few state variables will provide chance for PDE solution

So give it a try. . .

HJM (I)

$$df(t,T) = \sigma_f(t,T)\left(\int_t^T \sigma_f(t,s)ds\right)dt + \sigma_f(t,T)dW(t)$$

setting

$$\sigma_f(t,T) = h(t)g(T)$$

and integrating yields

$$f(t,T) = f(0,t) + \frac{g(T)}{g(t)}\left(x(t) + y(t)\frac{1}{g(t)}\int_t^T g(s)ds\right)$$

with

$$\begin{aligned} dx(t) &= \left(\frac{g'(t)}{g(t)}x(t) + y(t)\right)dt + g(t)h(t)dW(t) \\ dy(t) &= \left(g^2(t)h^2(t) + 2\frac{g'(t)}{g(t)}y(t)\right)dt \end{aligned}$$

HJM (II)

$$\begin{aligned} dx(t) &= \left(\frac{g'(t)}{g(t)}x(t) + y(t)\right)dt + g(t)h(t)dW(t) \\ dy(t) &= \left(g^2(t)h^2(t) + 2\frac{g'(t)}{g(t)}y(t)\right)dt \end{aligned}$$

with

$$\begin{aligned} \frac{g'(t)}{g(t)} &= -k(t) \\ g(t)h(t) &= \eta\big(t, x(t), y(t)\big) \end{aligned}$$

yields

$$\begin{aligned} dx(t) &= \big(y(t) - k(t)x(t)\big)dt + \eta\left(t, x, (t), y(t)\right)dW(t) \\ dy(t) &= \big(\eta^2\left(t, x(t), y(t)\right) - 2k(t)y(t)\big)dt \end{aligned}$$

Cheyette 1f (I)

See e.g. [1], [2]

$$
\begin{aligned}
dx(t) &= \big(y(t) - k(t)x(t)\big)dt + \eta(t, x(t), y(t))dW(t) \\
dy(t) &= \big(\eta^2(t, x(t), y(t)) - 2k(t)y(t)\big)dt \\
r(t) &= x(t) + f(0, t)
\end{aligned}
$$

Zero Coupon Bond:

$$
\begin{aligned}
P\big(t, T; x(t), y(t)\big) &= \frac{D(T)}{D(t)} \exp\left(-G(t,T)x(t) + \frac{1}{2}G^2(t,T)y(t)\right) \\
G(t,T) &= \int_t^T e^{-\int_t^u k(s)ds} du \;=\; \frac{1}{k}\left(1 - e^{-k(T-t)}\right)
\end{aligned}
$$

For any choice of $\eta\big(t, x(t), y(t)\big)$!
So, what is the best choice of $\eta\big(t, x(t), y(t)\big)$?

Cheyette 1f (II)

Displaced diffusion (DD)

$$\eta(t,x) = \big(m\,r(x,t) + (1-m)L\big)\sigma(t)$$
$$\eta(t,x,y) = \big(m\,S_{nN}(x,y,t) + (1-m)L\big)\sigma(t)$$

Constant Elasticity of Variance (CEV)

$$\eta(t,x) = r(x,t)^{\alpha}\sigma(t) \quad ; \quad \eta(t,x) = S_{nN}(x,y,t)^{\alpha}\sigma(t)$$

DD with stochastic volatility (SV)

$$\eta(t,x) = \big(m\,r(t) + (1-m)L\big)\sqrt{V(t)}\sigma(t)$$
$$dV = \beta(1-V(t))dt + \sqrt{V(t)}\epsilon dW_V \;\; ; \;\; V(0) = 1$$

Cheyette 1f (III)

Displaced Diffusion: $\eta(t,x) = \big(m\,r(t) + (1-m)L\big)\sigma(t)$
with $m = 0$

$$\begin{aligned}
\eta(t,x) &= \big(mr(t) + (1-m)L\big)\sigma(t) = L\sigma(t) \\
dx(t) &= \big(y(t) - kx(t)\big)dt + L\sigma(t)dW(t) \\
dy(t) &= \big(L^2\sigma^2(t) - 2ky(t)\big)dt
\end{aligned}$$

yields extended Vasicek model

$$\begin{aligned}
y(t) &= e^{-2kt}L^2\int_0^t \sigma^2(s)e^{-2ks}ds \\
dx(t) &= \big(y(t) - kx(t)\big)dt + L\sigma(t)dW(t)
\end{aligned}$$

Cheyette 1f (IV)

Displaced Diffusion: $\eta(t,x) = \big(m\,r(t) + (1-m)L\big)\sigma(t)$
with $m = 1$

$$\begin{aligned}
\eta(x,t) &= \big(mr(t) + (1-m)L\big)\sigma(t) = r(x,t)\sigma(t) \\
dx(t) &= \big(y(t) - kx(t)\big)dt + \eta(x,t)dW(t) \\
dy(t) &= \big(\eta^2(x,t) - 2ky(t)\big)dt = \big(x^2(t)\sigma^2(t) - 2ky(t)\big)dt
\end{aligned}$$

we get a kind of lognormal model

$$\begin{aligned}
dx(t) &= \big(y(t) - kx(t)\big)dt + r(x,t)\sigma(t)dW(t) \\
dy(t) &= \big(\eta^2(x,t) - 2ky(t)\big)dt = \big(\sigma^2(t)x^2(t) - 2ky(t)\big)dt
\end{aligned}$$

Intermediate Summary (I)

What do we have?

- shortrate model with only few state variables
- variable volatility specification (including stochastic volatility)
- analytic function for zero coupon bond: $P\big(t, T, x(t), y(t)\big)$

What do we need?

- fast calculation of instruments to calibrate to (e.g. swaptions)
- simulation of process
- scheme for PDE solving

Cheyette 1f Swaptions (I)

Displaced Diffusion:

$$\eta(t,x) = \big(mr(t) + (1-m)L\big)\sigma(t)$$

Swaption in swap-measure (annuity measure) [3]: $S_{nN}(t) = \frac{P(t,t_n)-P(t,t_N)}{\sum_1^N \alpha_i P(t,t_i)}$

$$\begin{aligned} dS_{nN} &= \frac{\partial S_{nN}}{\partial x}dx = \frac{\partial S_{nN}}{\partial x}\Big(mr(t) + (1-m)L\Big)\sigma(t)dW \\ &\approx \left[\frac{\partial S_{nN}}{\partial x}\frac{(mr(t)+(1-m)L)}{(mS_{nN}(t)+(1-m)L)}\right]_{x=y=0}\Big(mS_{nN} + (1-m)L\Big)\sigma(t)dW \\ dS_{nN} &\approx \big(mS_{nN} + (1-m)L\big)\lambda(t)dW \end{aligned}$$

Displaced Diffusion (lognormal in $\tilde{S}$):

$$\begin{aligned} \mathcal{SWP} &= PV_{01}\mathsf{E}\left[(S-K)^+\right] = \frac{PV_{01}}{m}\mathsf{E}\left[(\tilde{S}-\tilde{K})^+\right] = \frac{PV_{01}}{m}\mathsf{BS}(\tilde{S},\tilde{K}) \\ \text{with } \tilde{S} &= mS + (1-m)L \quad ; \quad \tilde{K} = mK + (1-m)L \end{aligned}$$

Cheyette 1f Swaptions (II)

Comments on approximation [3]:

$$S_{nN}(t) = \frac{P(t,t_n) - P(t,t_N)}{\sum_1^N \alpha_i P(t,t_i)} \quad ; \quad P(t,T) = \frac{D(T)}{D(t)} \exp\left(-G(t,T)x + \frac{1}{2} G^2(t,T) y \right)$$

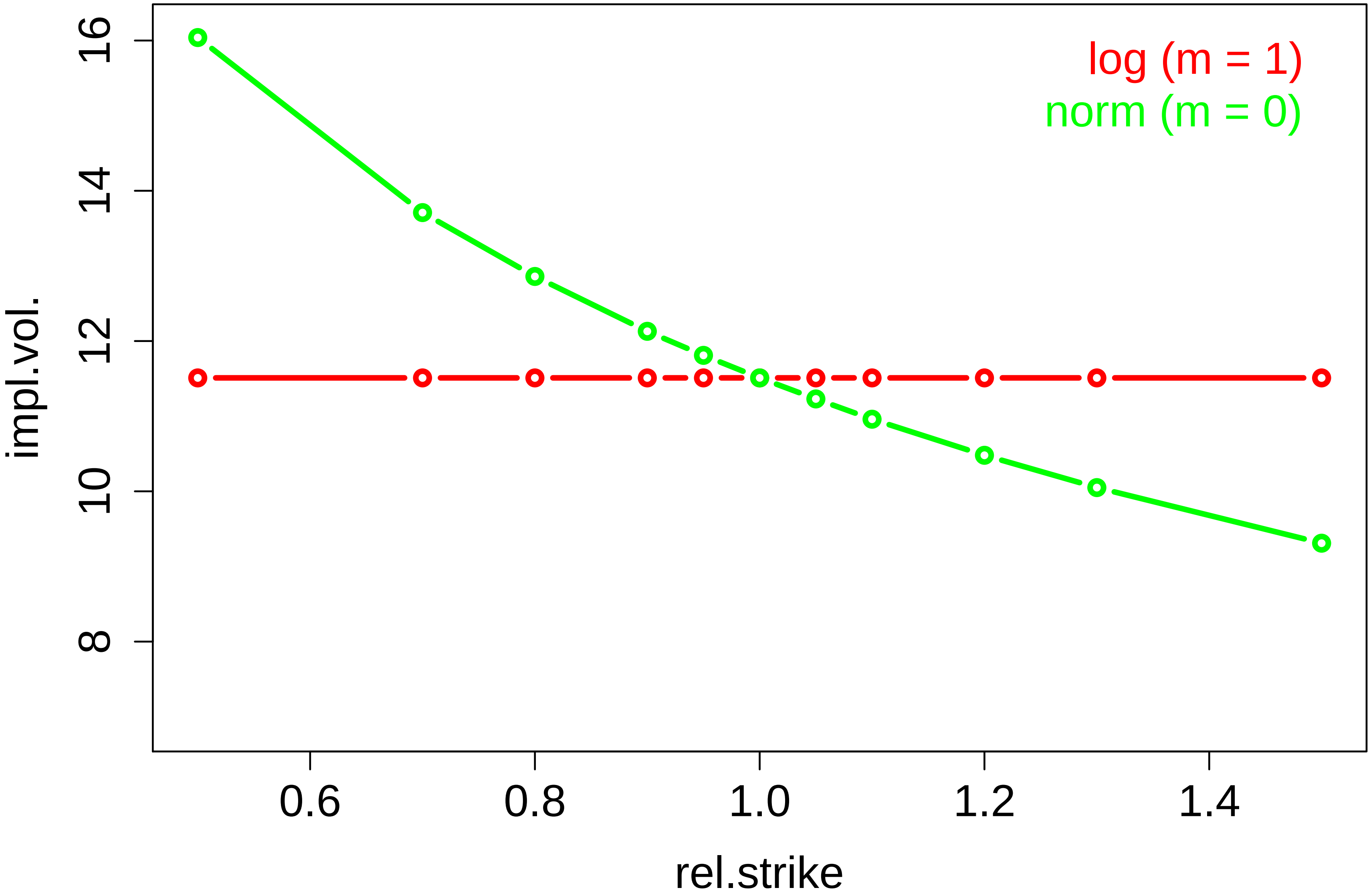
Errors in approximation

- freezing $\frac{\partial S}{\partial x} \approx \left[\frac{\partial S}{\partial x}\right]_{x=y=0}$ removes variance
- since $E\left[\frac{\partial S}{\partial x}\right] \neq E\left[\left[\frac{\partial S}{\partial x}\right]_{x=y=0}\right]$ expectation is biased

The above effects are small and partially offsetting.
(Unfortunately true only for $m \approx 0$.)

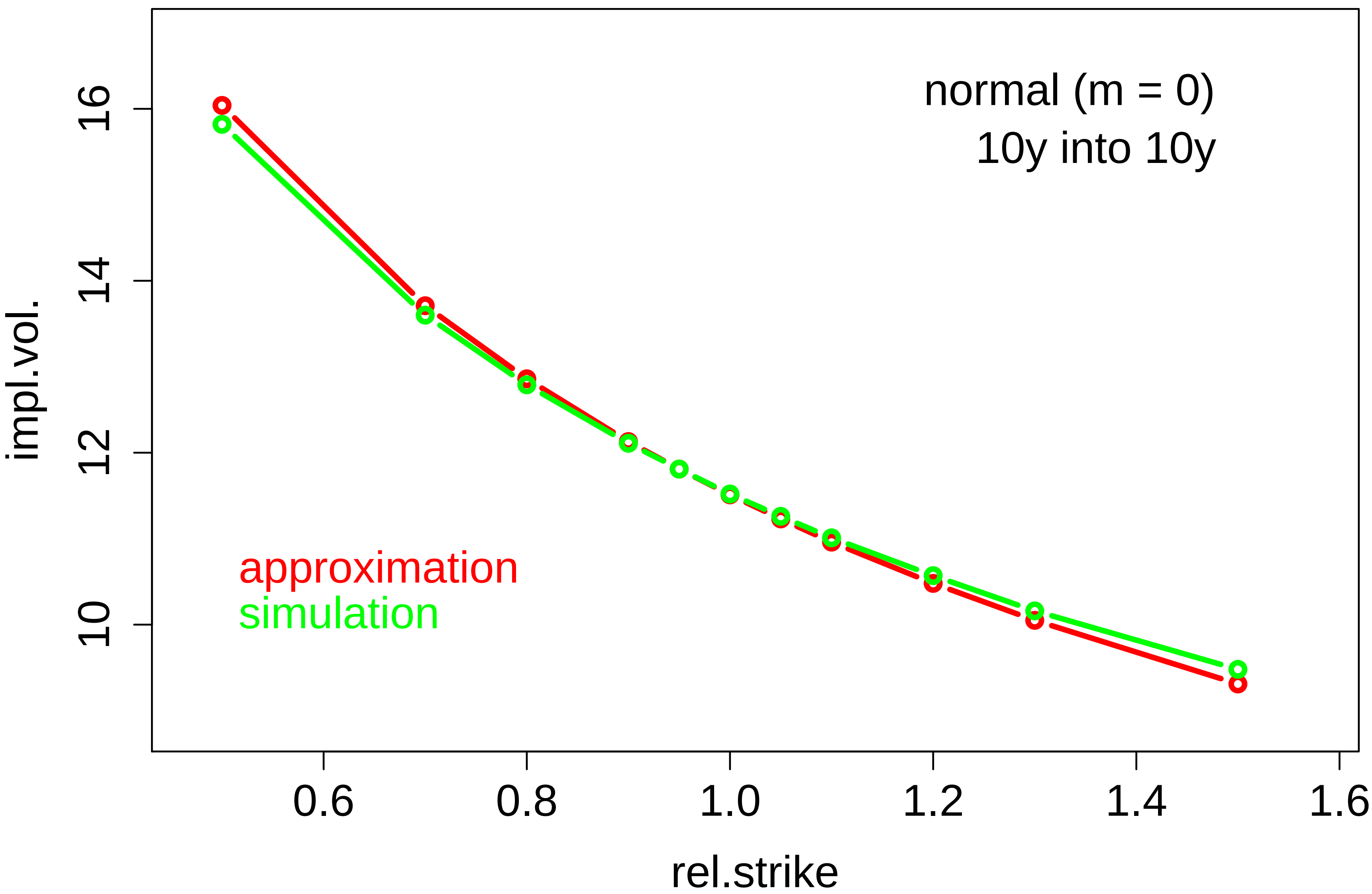
- no drift correction for $x(t)$ in annuity meassure
  $\Rightarrow$ residual skew and biased expectation

Cheyette 1f Swaptions (III)

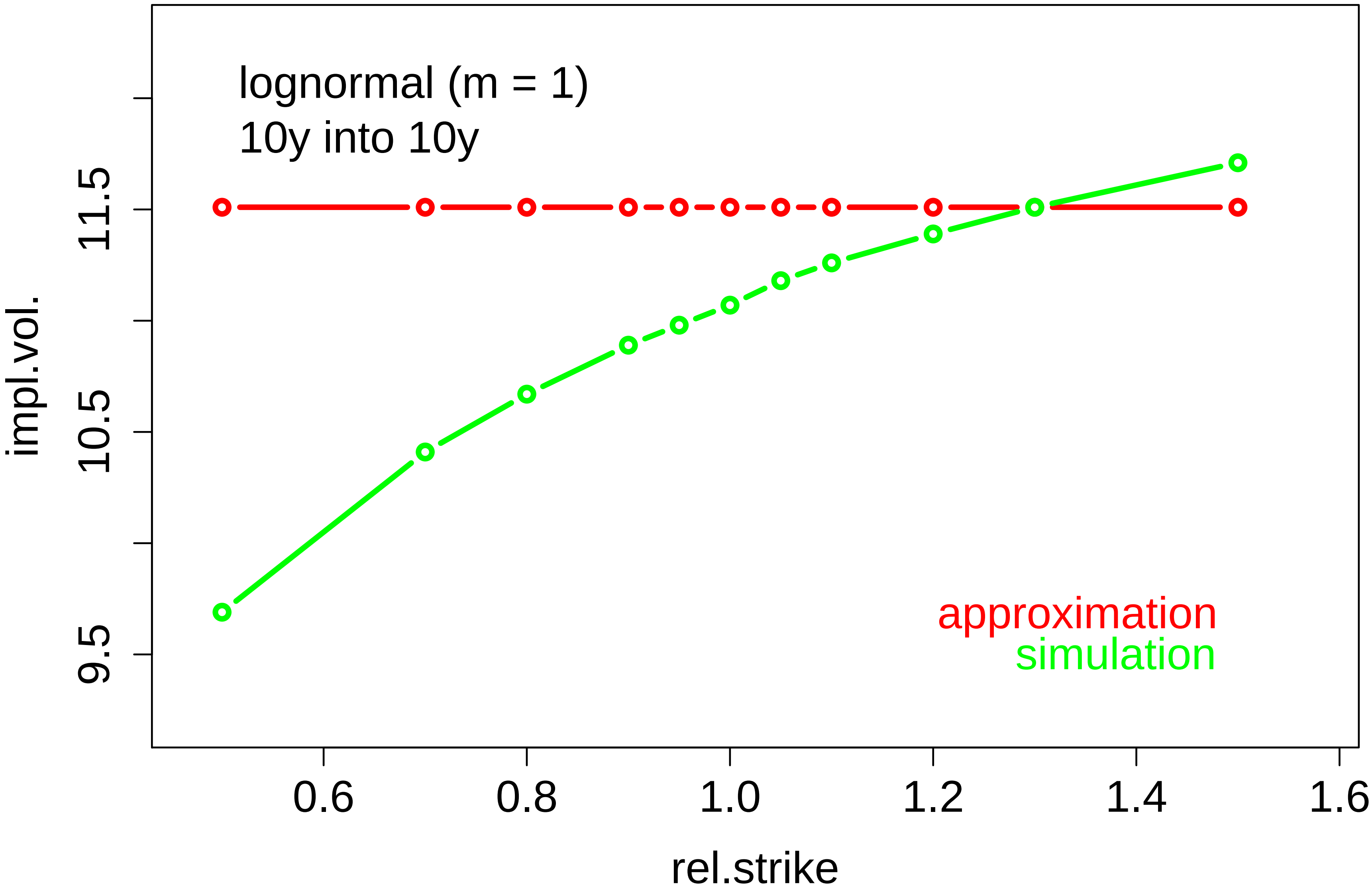


$$dS_{nN} = \big(mS_{nN} + (1-m)L\big)\lambda(t)dW$$

Cheyette 1f Swaptions (IV)



Cheyette 1f Swaptions (V)



Cheyette 1f stochastic volatility (SV)

SDE (see e.g. [2]):

$$
\begin{aligned}
dx(t) &= \big(y(t) - kx(t)\big)dt + \eta(x,t)\sqrt{V(t)}dW_x(t) \\
dy(t) &= \big(\eta^2(x,t) - 2ky(t)\big)dt \\
\eta(x,t) &= \big(m(t)x(t) + (1-m(t))L\big)\sigma(t) \\
dV(t) &= \beta\big(1 - V(t)\big)dt + \epsilon(t)\sqrt{V(t)}dW_V(t) \\
dW_x dW_V &= 0
\end{aligned}
$$

- Zero Coupon Bond: like in Cheyette without SV
(Unspanned Stochastic Volatility) Zero Coupon Bond:

$$
\begin{aligned}
P(t,T;x,y) &= \frac{D(T)}{D(t)} \exp\left(-G(t,T)x(t) + \frac{1}{2}G^2(t,T)y(t)\right) \\
G(t,T) &= \frac{1}{k}\left(1 - e^{-k(T-t)}\right)
\end{aligned}
$$

Cheyette 1f SV Swaption (I)

Start with time constant displacement (m):

$$\eta(x,t) = \big(mr(t) + (1-m)L\big)\sigma(t)$$
$$dV(t) = \beta\big(1 - V(t)\big)dt + \epsilon(t)\sqrt{V(t)}dW_V(t)$$

- change of measure into swap-measure (like in non-SV case)
- due to 0-correlation between dW_x and dW_V no change of drift in dV

Swaption in swap-measure (annuity measure) plus approximations:

$$dS_{nN} = \Big(mS_{nN} + (1-m)L\Big)\lambda(t)\sqrt{V(t)}dW$$
$$dV(t) = \beta\big(1 - V(t)\big)dt + \epsilon\sqrt{V(t)}dW_V(t)$$

- well known Heston SDE (+ Displaced Diffusion)

Cheyette 1f SV Swaption (II)

$$\begin{aligned} dS_{nN} &= \Big(mS_{nN} + (1-m)L\Big)\lambda(t)\sqrt{V(t)}dW \\ dV(t) &= \beta\big(1 - V(t)\big)dt + \epsilon(t)\sqrt{V(t)}dW_V(t) \end{aligned}$$

$$\begin{aligned} \mathcal{SWP} &= PV_{01}\mathsf{E}\left[(S-K)^+\right] = \frac{PV_{01}}{m}\mathsf{E}\left[(\tilde{S}-\tilde{K})^+\right] = \frac{PV_{01}}{m}\text{Heston}(\tilde{S},\tilde{K}) \\ \text{with }\ \tilde{S} &= mS + (1-m)L \quad ; \quad \tilde{K} = mK + (1-m)L \end{aligned}$$

The Heston part can be solved by

- fundamental transform (see below)
- volatility expansion

Cheyette 1f SV Swaption (III)

$$\text{Heston}(\tilde{S},\tilde{K}) = \tilde{S}(0) + \frac{\tilde{K}}{2\pi}\Re\left[\int_{ik_i-\infty}^{ik_i+\infty}\frac{e^{-i\omega\ln\frac{\tilde{S}(0)}{\tilde{K}}}}{\omega^2-i\omega}e^{a(0,\omega)+b(0,\omega)}d\omega\right]$$

$$\stackrel{k_i=0.5}{=} \tilde{S}(0) + \frac{\tilde{K}}{\pi}\int_0^\infty \frac{\sqrt{\frac{\tilde{S}(0)}{\tilde{K}}}\cos\left(-k\ln\frac{\tilde{S}(0)}{\tilde{K}}\right)}{k^2+0.25}e^{a(0,k+0.5i)+b(0,k+0.5i)}dk$$

$$\frac{da}{dt} = -\beta b \quad ; \quad a(T,\omega)=0$$

$$\frac{db}{dt} = -\beta b - \frac{1}{2}\epsilon^2(t)b^2 + \frac{1}{2}\lambda^2(t)m^2k^2 \quad ; \quad b(T,\omega)=0$$

Riccati ordinary differential equation has

- analytical solution for ϵ and λ constant,
- piecewise analytical solution for ϵ and λ piecewise constant and
- numerical solution for $\epsilon(t)$ and $\lambda(t)$ (which is slower)

Unfortunately $\lambda(t)$ is **not** piecewise constant; (whereas $\sigma(t)$ is)

Cheyette 1f SV Swaption (IV)

- can solve model with $\lambda(t)$ and $\epsilon(t)$ and m
- **cannot** use lognormal property of Displaced Diffusion anymore when m becomes $m(t)$

$$
\begin{aligned}
dS_{nN} &= \big(m(t)S_{nN} + (1 - m(t))L\big)\lambda(t)\sqrt{V(t)}dW_S \\
dV(t) &= \beta\big(1 - V(t)\big)dt + \epsilon(t)\sqrt{V(t)}dW_V
\end{aligned}
$$

- find time averaged variables to approximate dynamics:
$\bar{m}$ for solvebility ; $\bar{\lambda}$ and $\bar{\epsilon}$ for speed up

$$
\begin{aligned}
d\bar{S}_{nN} &= \big(\bar{m}\bar{S}_{nN} + (1 - \bar{m})L\big)\bar{\lambda}\sqrt{V(t)}dW_{\bar{S}} \\
d\bar{V}(t) &= \beta\big(1 - \bar{V}(t)\big)dt + \bar{\epsilon}\sqrt{\bar{V}(t)}dW_{\bar{V}}
\end{aligned}
$$

Cheyette 1f SV Swaption (V)

Average volatility of variance $\bar{\epsilon}$ (V.Piterbarg[5]):

Equate variance of realized volatility

$$E\Big[\Big(\int_0^T \lambda^2(t)\bar{V}(t)dt\Big)^2\Big] = E\Big[\Big(\int_0^T \lambda^2(t)V(t)dt\Big)^2\Big]$$

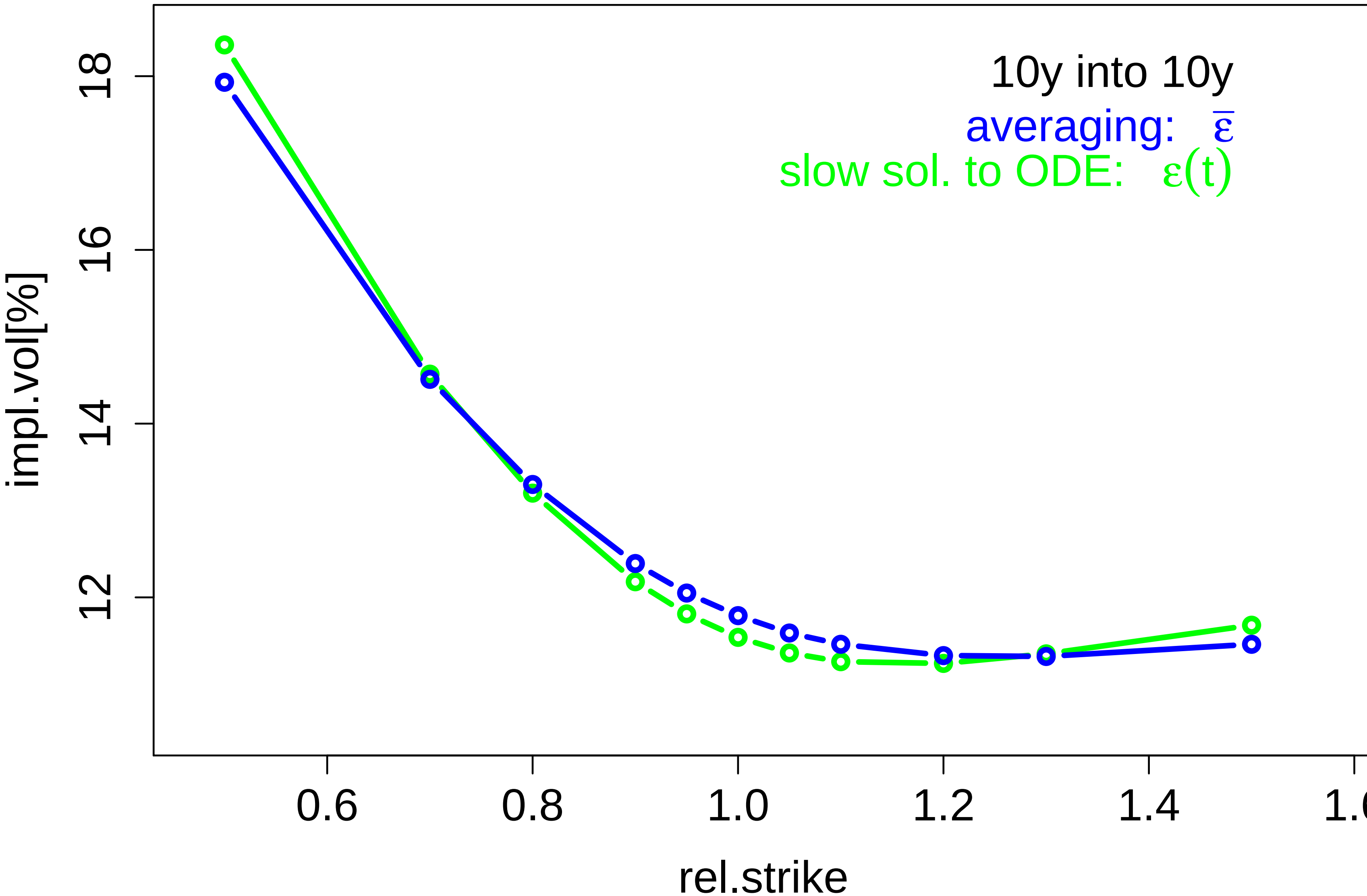
yielding

$$\bar{\epsilon}^2 = \frac{\int_0^T \epsilon^2(t)w(t)dt}{\int_0^T w(t)dt}$$

with

$$w(t) = \int_t^T \int_s^T \lambda^2(u)\lambda^2(s)e^{\beta(u-s)}e^{2\beta(s-t)}\,du\,ds$$

Cheyette 1f SV Swaption (VI)



$L = 4\%$ $k = 2.4\%$ $m = 0.2$ $\sigma = 18.5\%$ $\epsilon(0) = 80\%, \epsilon(10) = 20\%$

Cheyette 1f SV Swaption (VII)

Effective skew parameter $\bar{m}$ (V.Piterbarg[5]):

$$dS_{nN} = \eta\left(S(t),t\right)\lambda(t)\sqrt{V(t)}dW_S \quad \text{with} \quad \eta\left(S(t),t\right) = m(t)S_{nN} + (1-m(t))L$$

$$dV(t) = \beta\left(1-V(t)\right)dt + \epsilon(t)\sqrt{V(t)}dW_V$$

V.P. shows that

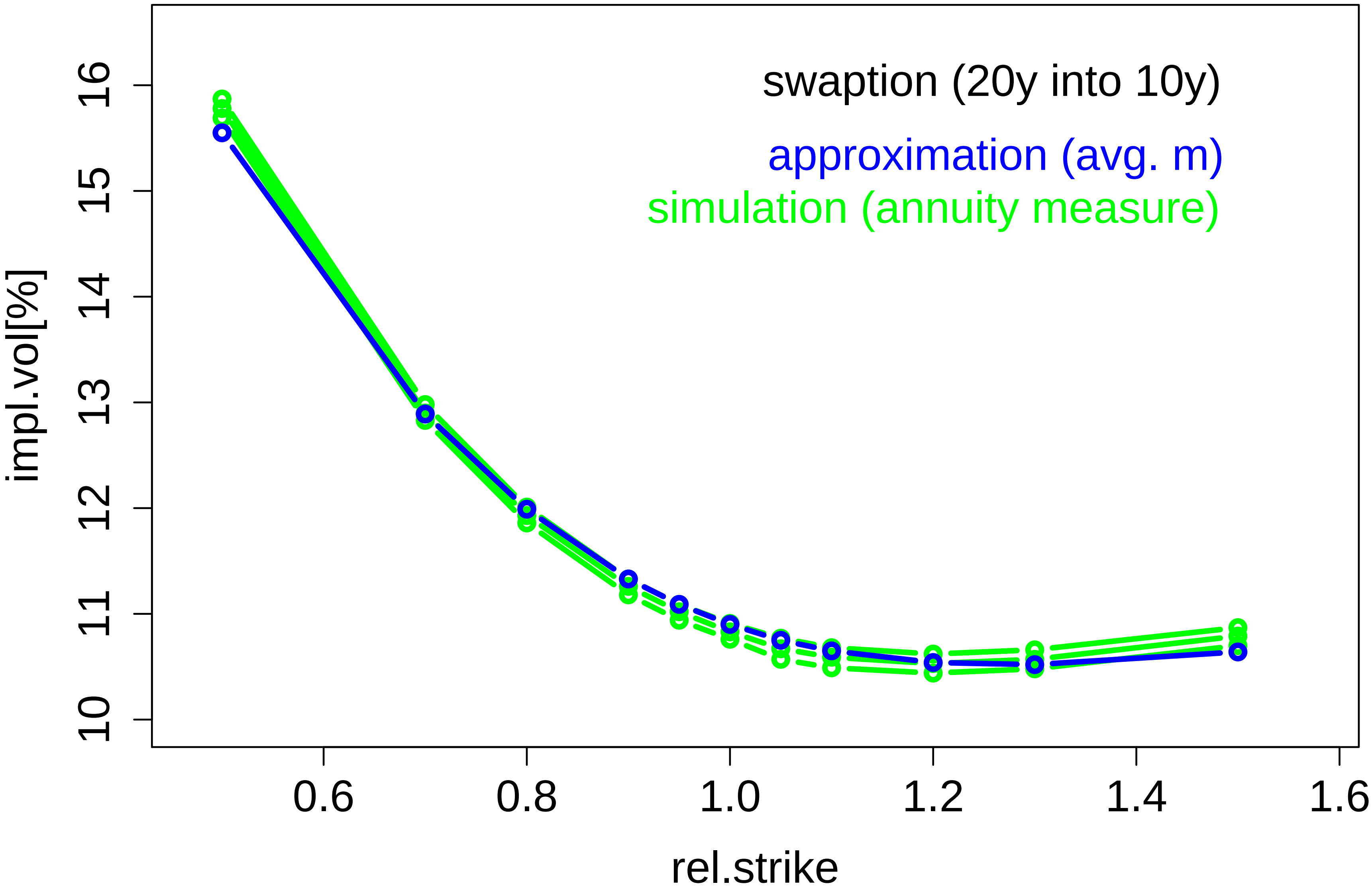
$$\bar{\eta}^2(S(t)) = \frac{\int_0^T w(t)\eta(S(t),t)\lambda^2(t)dt}{\int_0^T w(t)\lambda^2(t)dt} \quad \text{with} \quad w(t) = E\Big[V(t)\big(S(t)-S(0)\big)^2\Big]$$

minimizes the difference of the second and third moment between the time averaged and timedependent SDE.

For the given $\eta\left(S(t),t\right)$ this yields

$$\bar{m} = \frac{\int_0^T m(t)w(t)\lambda^2(t)dt}{\int_0^T w(t)\lambda^2(t)dt} \quad \text{with} \quad w(t) = \int_0^t \lambda^2(s)ds + \bar{\epsilon}^2 e^{-\beta t}\int_0^t \frac{\lambda^2(s)}{2\beta}\left(e^{\beta s}-e^{-\beta s}\right)ds$$

Cheyette 1f SV Swaption (VIII)



$L = 4\%$ $k = 2.4\%$ $m(0) = 0.8, m(15) = 0.3$ $\sigma = 18.5\%$ $\epsilon = 60\%$

Cheyette 1f SV Swaption (IX)

Effective volatility $\bar{\lambda}$ (V.Piterbarg[5]):

step 1: Black-Scholes ATM-Swaption formula as function of variance

$$g(x) = \frac{S_{nN}}{\bar{m}} \left(2N(\frac{1}{2}\bar{m}\sqrt{x}) - 1 \right)$$

is approximated with

$$\tilde{g}(x) = a + be^{-cx}.$$

by setting

$$g(\xi) = \tilde{g}(\xi) \;\; ; \;\; g'(\xi) = \tilde{g}'(\xi) \text{ and } g''(\xi) = \tilde{g}''(\xi)$$

step 2: Equate expectations for time dependent and time averaged λ

$$\begin{aligned} E\Big[\tilde{g}\Big(\int_0^T \lambda^2(t) V(t) \Big)\Big] &= E\Big[\tilde{g}\Big(\bar{\lambda}^2 \int_0^T V(t)\Big)\Big] \\ a + bE\Big[e^{-c\int_0^T \lambda^2(t)V(t)dt}\Big] &= a + bE\Big[e^{-c\bar{\lambda}^2 \int_0^T V(t)dt}\Big] \end{aligned}$$

Cheyette 1f SV Swaption (X)

still step 2:

$$
\begin{aligned}
a + bE\left[e^{-c\int_0^T \lambda^2(t)V(t)dt}\right] &= a + bE\left[e^{-c\bar{\lambda}^2\int_0^T V(t)dt}\right] \\
E\left[e^{-c\int_0^T \lambda^2(t)V(t)dt}\right] &= E\left[e^{-c\bar{\lambda}^2\int_0^T V(t)dt}\right] \\
e^{A(0,T)+B(0,T)} &= e^{\bar{A}(0,T)+\bar{B}(0,T)}
\end{aligned}
$$

$a()$ and $a()$ follow Riccati ODE. For $\bar{a}()$ and $\bar{a}()$ anayltic solutions exist.

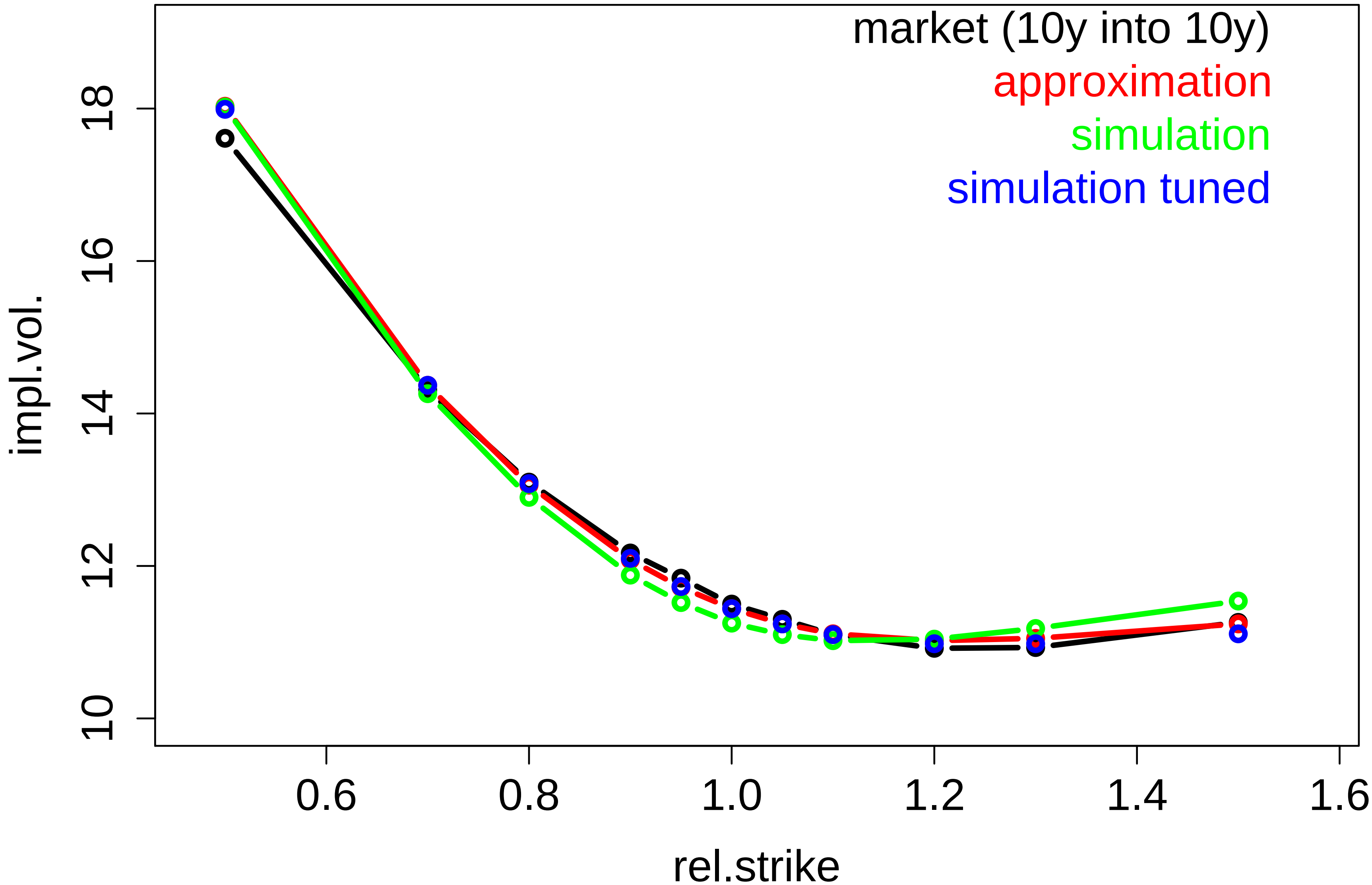
- time dependent part: once solve for $a()$ and $b()$ numerically
- time independent part: solve for $\bar{\lambda}$ numerically (cheap since analytical solutions to $\bar{a}()$ and $\bar{b}()$ exist

Cheyette 1f SV Swaption (XI)

Some empirical observations for swaptions in the Euro market in June:

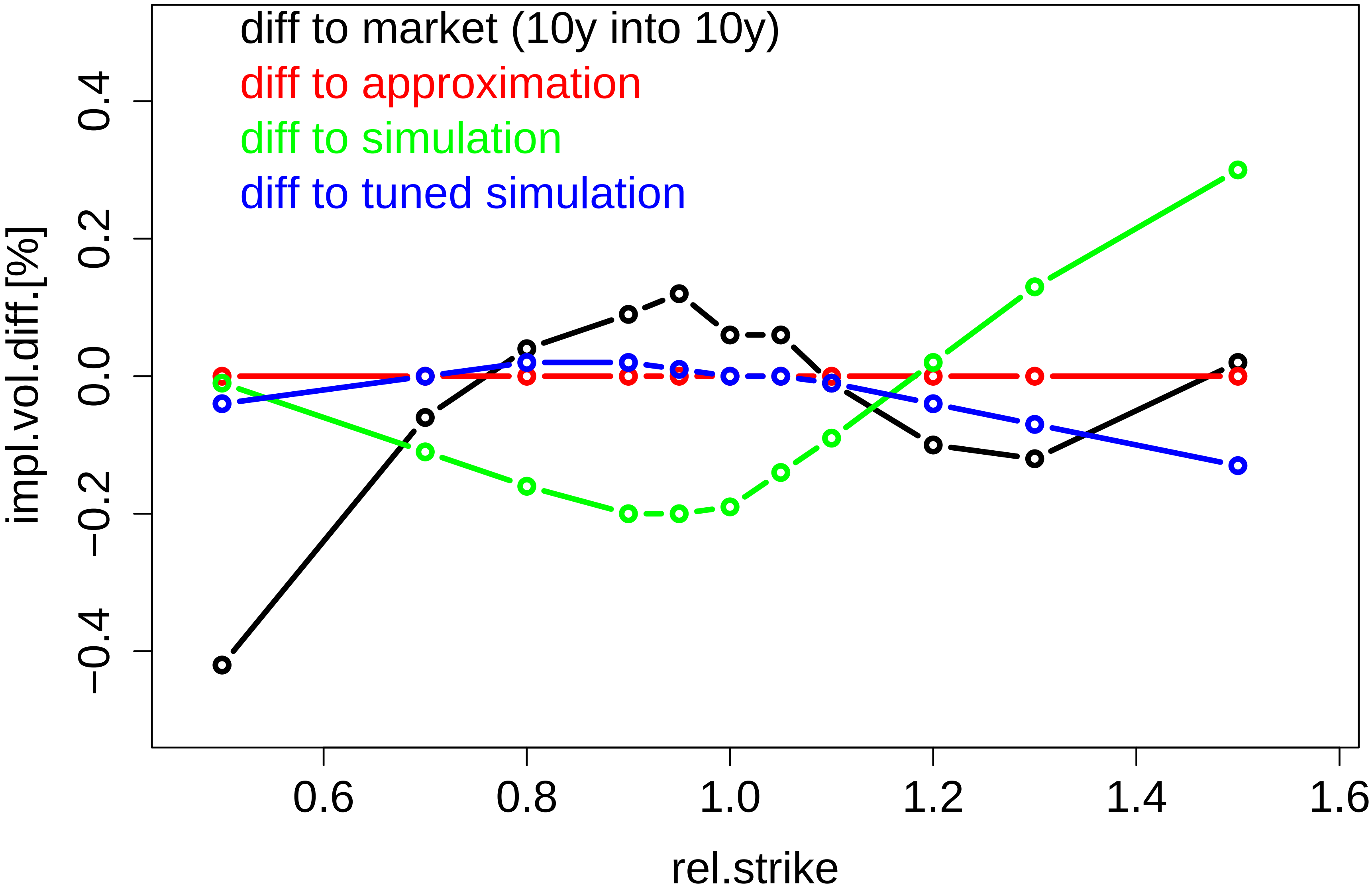
- the skew parameter m tends to be well below 1 and not strongly varying
- the smile parameter ϵ is constant around 50% for larger t,
  a bit larger (60%) at small t ($t < 2$years)
- the volatiliy $\sigma(t)$ is about 18% and not strongly fluctuating

Cheyette 1f SV Swaption (XIIa)



$L = 4\% \;\; k = 2.4\% \;\; m \approx 0.2[0.1] \;\; \sigma \approx 18\% \;\; \epsilon = 60\%[50\%]$

Cheyette 1f SV Swaption (XIIb)



$L = 4\%$ $k = 2.4\%$ $m \approx 0.2[0.1]$ $\sigma \approx 18\%$ $\epsilon = 60\%[50\%]$

Intermediate Summary (II)

- by
  - transformation into swap measure
  - freezing of $\frac{dS_{nN}}{dx}\eta(r) \approx \left[\frac{dS_{nN}}{dx}\frac{\eta(r)}{\eta(S_{nN})}\right]_{x=y=0}\eta(S_{nN})$
  - using fact that displaced diffusion leads to lognormal distribution in substitute variable
  - time averaging of $\sigma(t)$, $\lambda(t)$ and $\epsilon(t)$

  we get more (m small) or less (m large) accurate numerical solution
- can (largely) remove residual skew of approximation by correting for deviation between approximation and simulation once (denoted by tuned simulation)
- hence we are prepared for calibration of the model

Calibration to Swaptions

- minimize sum of weighted squared relative deviations of model price from market price

  global: for all model parameters at once : slow and unstable
  stepwise: assume knowledge of time independent parameters k and β;
  fit time dependent parameters $m(t)$, $\sigma(t)$ and $\epsilon(t)$ for subsequent swaption expiries
  iterative: optimize time independent parameters combined with a stepwise calibration for each valuation of the cost function

- depending of the number of factors, time intervals and market quotes this takes from fractions of a second to several minutes

- for very accurate calibration one intermediate simulation step is helpful (removing residual skew in swaption price approximation for $m > 0$)

Simulation of Variance Process (CIR)

$$
\begin{aligned}
dV(t) &= \beta\big(1 - V(t)\big)dt + \epsilon\sqrt{V(t)}dW_V(t) \qquad V(0) = 1 \\
V(t + \Delta t) &= V(t) + \beta\big(1 - V(t)\big)\Delta t + \epsilon\sqrt{V(t)}\sqrt{\Delta t}Z
\end{aligned}
$$

- process itself **cannot** yield $V(t) < 0$
- for large β and / or large ϵ : Euler discretization **can** yield negative $V(t)$
  - if negative, set to 0 : does not work at all
  - if negative, mirror to postive value : does not work well
  - sample from non-central χ^2 [6]: works well, no antithetics (fast implementation in GSL (not inverting distribution function))
  - moment matched log-normal[7] : works well, antithetics possible

$$
\begin{aligned}
V(t + \Delta t) &= \big(1 + (V(t) - 1)e^{-\beta\Delta}\big)e^{-0.5\Gamma^2(t)+\Gamma(t)Z} \\
\Gamma^2(t) &= \ln\left(1 + \frac{\epsilon^2 V(t)(1 - e^{-2\beta\Delta t})}{2\beta(1 + (V(t) - 1)e^{-\beta\Delta t})^2}\right)
\end{aligned}
$$

PDE solution (I)

PDE for 1factor without stochastic volatility

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= (D_x + D_y - r)V \\
D_x &= (y - kx)\frac{\partial}{\partial x} + \frac{1}{2}\Big(mr(t) - (1-m)L\Big)^2 \frac{\partial^2}{\partial x^2} \\
D_y &= \left(\Big(mr(t) - (1-m)L\Big)^2 - 2ky\right)\frac{\partial}{\partial y}
\end{aligned}
$$

- two dimensional PDE; y-dimension is convection only!
  - ADI
  - Craig Sneyd (splitting scheme)
  - IMEX (see below)

PDE solution (II)

IMEX-scheme [8]:

$$
\begin{aligned}
\frac{V(t^+) - V(t)}{\Delta t} \quad = \quad & \frac{\mu_y}{\Delta y}(V(y^+,t) - V(t)) \quad \text{alt.}\Big[\frac{\mu_y}{\Delta y}\Big(-\frac{1}{6}V(y^{--},t) + V(y^-,t) - \frac{1}{2}V(t) - \frac{1}{3}V(y^+,t)\Big)\Big] \\
+ \quad & \frac{1}{2}\left(\frac{1}{2}\sigma_x^2 \frac{V(x^+,t) + V(x^-,t) - 2V(t)}{\Delta x^2} + \mu_x \frac{V(x^+,t) - V(x^-,t)}{2\Delta x} - rV(t)\right) \\
+ \quad & \frac{1}{2}\left(\frac{1}{2}\sigma_x^2 \frac{V(x^+,t^+) + V(x^-,t^+) - 2V(t^+)}{\Delta x^2} + \mu_x \frac{V(x^+,t^+) - V(x^-,t^+)}{2\Delta x} - rV(t^+)\right)
\end{aligned}
$$

- purely explicit in y
- use upwind scheme in y, not central differences : oscillations
- use third order upwind in y to increase accuracy to $O(\Delta y^3)$: no penalty in solving system of linear equations since explicit

- can get accurate solution of swaption with 15 points in y
(number of necesarry gridpoints in y depends on m; for $m \approx 0$ we can get away with even less than 15)

Multifactor

One multidimensional extension of the model is straight forward

$$
\begin{aligned}
dx_i(t) &= \big(y_{ij}(t) - k_i x_i(t)\big)dt + \sum_{j=1}^{f} \eta_{ij}(t) dW_j(t) \;\; ; \;\; dW_i dW_j = 0 \\
dy_{ij}(t) &= \big(\sum_{k=1}^{f} \eta_{ik}(t)\eta_{jk}(t) - (k_i + k_j)y_{ij}(t)\big)dt \\
\eta_{ij}(t, x_i, y_{ij}) &= \sigma_{ij}(t)\big(m(t)r(t) + (1 - m(t))L\big) \\
\sigma_{ij}(t) &= \text{lower triangular matrix} \\
r(t) &= \sum_i x_i(t) + f(0, t)
\end{aligned}
$$

- f stochastic and $f\frac{3+f}{2}$ total number of state variables
- analytic formula for P(t, $\vec{x}$, $\bar{y}$)
- similar approximations for swaption

Summary / Outlook

- What we have:
  - class of shortrate models with flexible volatility function
    - $*$ flexible smile and / or skew
    - $*$ stochastic volatility with corresponding dynamics
  - low dimensional Markow and analytical solution for ZCB:
    $\Rightarrow$ relatively cheap in MC-simulation and PDE
  - 1 factor version is capable of reproducing the smile for one underlying per expiry nicely (e.g coterminal swaptions); even swaptions not used in calibration process often still ok; further improvements by multi factor version expected
- Future plans:
  - improve swaption approximation in annuity measure for $m \neq 0$
  - extensions and checks of multi factor model
  - include jump diffusion

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